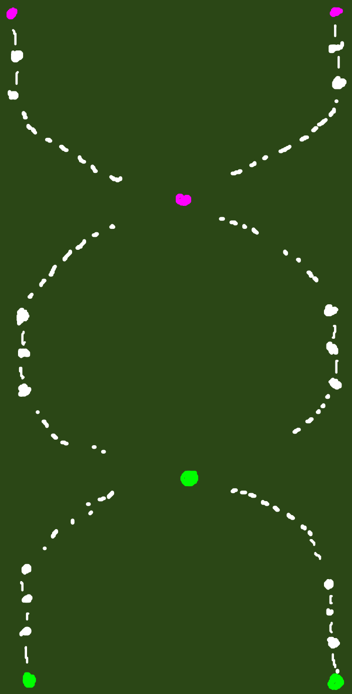
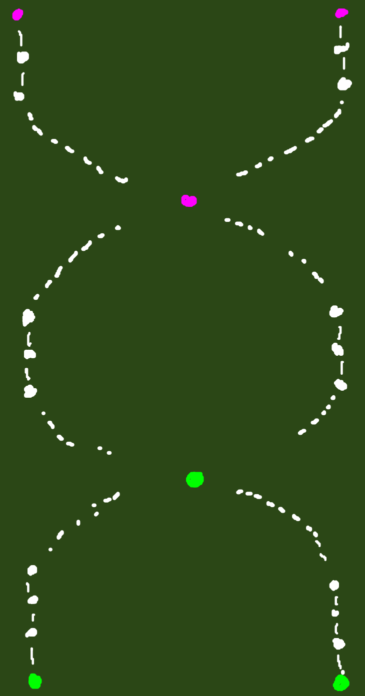


Priestley duality for MV algebras and beyond

BLAST
June 13, 2021

Stone-Priestley session

Sam v. Gool, joint work with Wesley Fussner, Mai Gehrke and Vincenzo Marra



Priestley dual space of the algebra of $\mathbb{Z}$-PL functions $[0,1] \to [0,1]$ localized at $1/2$

Forum Math. (2021),
arXiv: 2002.12715v3

Plan

1. Priestley duality extended to double quasi-operators

2. Applying correspondence theory

3. The case of MV-algebras

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   → How to dualize operations of implication type that respect **both** $\wedge$ and $\vee$?

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   → How to dualize equational axioms?

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   → How to dualize operations of implication type that respect <u>both</u> $\wedge$ and $\vee$?

2. Applying correspondence theory

   → How to dualize equational axioms?

3. The case of MV-algebras

   → What does this yield in the special case of MV?

1. Priestley duality extended to double quasi-operators

Def. Let L a bounded distributive lattice.

A double quasi-operator of reverse implication type, or, a "minus", is a binary function $\ominus : L \times L^{op} \to L$ satisfying the equations:

$-\ominus b$ is a hom. except for 1

$$\begin{cases} (a_1 \vee a_2) \ominus b = (a_1 \ominus b) \vee (a_2 \ominus b) \\ (a_1 \wedge a_2) \ominus b = (a_1 \ominus b) \wedge (a_2 \ominus b) \\ 0 \ominus b = 0 \end{cases}$$

$$\left.\begin{array}{l} a \ominus (b_1 \vee b_2) = (a \ominus b_1) \wedge (a \ominus b_2) \\ a \ominus (b_1 \wedge b_2) = (a \ominus b_1) \vee (a \ominus b_2) \\ a \ominus 1 = 0 \end{array}\right\}$$

$a \ominus -$ is an anti-hom except for $0 \leftrightarrow 1$

A pair $(L, \ominus)$ is a $\ominus$-algebra.

(Throughout the talk, we moreover assume $a \ominus 0 = a$, but this is inessential for the theory to work.)

Theorem (Gehrke–Jónsson) L a bounded DL with Priestley space X.

Any minus operator admits two canonical liftings

$$\ominus^\sigma, \ominus^\pi : \mathcal{D}X \times (\mathcal{D}X)^{op} \longrightarrow \mathcal{D}X$$

to minus operators on $\mathcal{D}X$,

and any equation on $(L, \ominus)$ lifts to both $(\mathcal{D}X, \ominus^\sigma)$ and $(\mathcal{D}X, \ominus^\pi)$.

But: $\ominus^\sigma$ respects $\bigvee$, while $\ominus^\pi$ respects $\bigwedge$ in both coordinates.

Example. $L \models a \wedge b \leq a \ominus (a \ominus b)$ $\approx$ "MV6"

implies

$\mathcal{D}X \models a \wedge b \leq a \ominus^\pi (a \ominus^\pi b)$.

Recall $X \overset{\mu}{\hookrightarrow} \mathcal{D}X$ via $\mu_x := X \setminus \uparrow x$.

$$M = \operatorname{im}(\mu).$$

For any $y \in X$, $- \ominus^{\sigma} \mu y : \mathcal{D}X \longrightarrow \mathcal{D}X$ preserves $\bigvee$, and therefore has an upper adjoint $- + \mu y : \mathcal{D}X \longrightarrow \mathcal{D}X$ defined by

$$u \ominus^{\sigma} \mu y \leq v \iff u \leq v + \mu y.$$

$\Diamond(\_,\_)$

$\wr$

Fact. For any $x, y \in X$, $\mu x + \mu y \in M \cup \{1\}$. $\quad + \rightsquigarrow R(\_,\_,\_)$

Recall $X \overset{\nu}{\hookrightarrow} \mathcal{D}X$ via $\nu x := \downarrow x$.

$$\mathcal{J} = \mathrm{im}(\nu).$$

For any $y \in X$, $- \ominus^{\pi} \nu y : \mathcal{D}X \longrightarrow \mathcal{D}X$ preserves $\overset{\text{non-}\emptyset}{\bigwedge}$, and therefore has a p. lower adjoint $- * \nu y : \mathcal{D}X \to \mathcal{D}X$ defined by

$$u \ominus^{\pi} \nu y \geqslant v \iff \left(u \geqslant v * \mu y \text{ and } u \leq 1 \ominus^{\pi} \nu y \right).$$

<u>Fact</u>. For any $x, y \in X$, $\nu x * \nu y \in \mathcal{J} \cup \{0\}$, if $\nu x \leq 1 \ominus^{\pi} \nu y$.

Both $+$ and $*$ are partial, but we avoid mention of their domains here, see paper for full details.

Thus: $\ominus^{\sigma}$ yields $+$ on the dual,
$\ominus^{\pi}$ yields $*$ on the dual.

Prop $(*+)$ $\quad x * y = \inf \{ x + w : w \not\leq y \}$

→ How does $+$ "feel" that it is dual to a double minus?

Prop $(+)$ For any $x \in X$, the translation $x + -$ has a totally ordered image and an upper adjoint, $k(x, -)$.

Note: N. Martínez previously found the function $x \mapsto k(x,x)$ "in the wild" in the 90s. (for MV-algebras)

Theorem (Extended Priestley duality for $\ominus$-algebras.)

The category of $\ominus$-algebras is dually equivalent to a category of **$\ominus$-spaces**:

$$(X, i, +, *) \quad \text{where}$$

i records what $a \mapsto 1 \ominus a$ does.

X a Priestley space,

$i : X^{op} \to X$ cts. order-reversing function,

$+ : X^2 \nrightarrow X$ upper cts. order-preserving partial fn.,

$* : X^2 \nrightarrow X$ lower cts. order-preserving partial fn.,

$(*+)$ and $(+)$ hold.

effect algebras.

$R \subseteq X^3$

2. Applying Correspondence Theory

Building on the foundation of our duality for $\ominus$-algebras, we may now add axioms to obtain dualities for subvarieties.

Convenient notations :

$$\neg a := 1 \ominus a$$

$$a \oplus b := \neg(\neg a \ominus b).$$

Example axioms: (1) $\neg\neg a = a$

(2) $a \oplus b = b \oplus a$

(3) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(4) $\oplus$ is upper residual of $\ominus$.

$(a \oplus b) \ominus b \leq a$ and $a \leq (a \ominus b) \oplus b$.

$$\boxed{\begin{array}{ll} (1) & \neg\neg a = a \\ (2) & a \oplus b = b \oplus a \\ (3) & (a \oplus b) \oplus c = a \oplus (b \oplus c). \\ (4) & \oplus \text{ is upper residual of } \ominus . \end{array}}$$

For each of these examples, if a $\ominus$-algebra L satisfies the equation (n) then so does $(\mathcal{D}X, \ominus^{\sigma})$.

Moreover, (4) implies that $\oplus^{\pi}$ <u>coincides with</u> $+$ (where defined).

Now a simple and standard application of correspondence theory yields duals for these axioms.

Theorem. A $\ominus$-algebra L satisfies

(1) $\neg\neg a = a$

(2) $a \oplus b = b \oplus a$

(3) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(4) $\oplus$ is upper residual of $\ominus$.

if and only if

its dual $\ominus$-space $(X, i, +, *)$ satisfies

(1) $i^2 = i$

(2) $+$ commutative

(3) $+$ associative

$\Big\}$ (when defined)

(4) i respects translation by $+$:

$x + y \leq z$ iff $i(z) + y \leq i(x)$.

$\Big\}$ pre-MV-space

3. The case of MV-algebras

(or: the use of $*$)

Fact. A $\ominus$-algebra is (term-equivalent to) an MV-algebra iff it satisfies (1)–(4) and $a \wedge b \leq a \ominus (a \ominus b)$ (MV6).

As we remarked in the beginning, MV6 lifts to

$$\mathcal{D}X \models u \wedge v \leq u \ominus^{\pi} (u \ominus^{\pi} v)$$

or equivalently

$$\mathcal{D}X \models \neg u \wedge v \leq (u \oplus^{\pi} v) \ominus^{\pi} u .$$

$\forall u, v$

Now, standard correspondence arguments let us rewrite this into

$$\forall x, y \in X, \quad \nu x \leq \neg \nu y \Rightarrow \nu x \leq (\nu x \underline{\oplus}^{\pi} \nu y) \underline{\ominus}^{\pi} \nu y.$$

To isolate the blue term, we must do:

$$\forall x, y \in X, \quad \nu x \leq \neg \nu y \Rightarrow \nu x * \nu y \leq \nu x \oplus^{\pi} \nu y.$$

This is where $*$ appears.

The condition can then be further rewritten to replace $\oplus^{\pi}$ by $+$, using standard methods.

Theorem. The category of MV-algebras is dually equivalent to the full subcategory of $\ominus$-spaces on MV-spaces, i.e., pre-MV-spaces satisfying the additional (first-order!) property:

$\forall (x,y) \in \mathrm{dom}(*)$, $x', y' \in X$, if $y' \nleq y$ and $x' + y' \leq x * y$ then $x' \leq x$.

"mixed cancellativity of $*$ and $+$" ?

Take-aways and directions

- Extended Priestley duality with partial functions (instead of relations) on the dual is possible in the general setting of $\ominus$-algebras.

- Correspondence & canonicity on top of this yields concrete results for subvarieties like MV.

  (Also see my first paper with Gehrke & Marra '14)

- Other example applications: - MTL-algebras (an example in the paper)
  - residuated lattices of Scott-continuous functions on complete chains

  (cf. recent work with Gualto, Metcalfe & Santschi).

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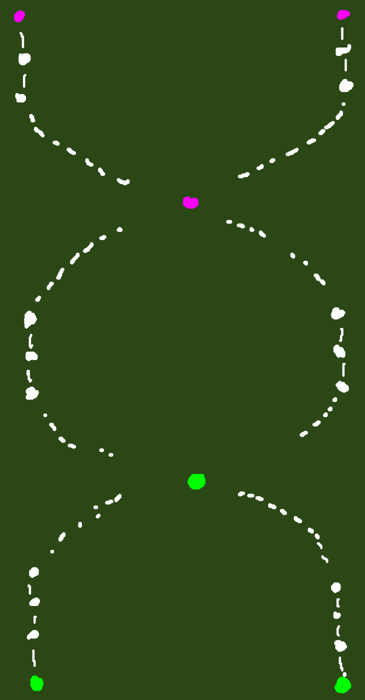

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