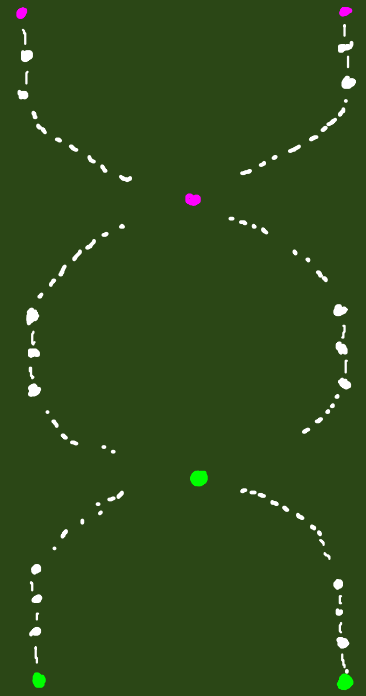


# Priestley duality for MV algebras and beyond

BLAST  
June 13, 2021

Stone-Priestley session

Sam v. Gool, joint work with Wesley Fussner, Mai Gehrke and Vincenzo Marra



Forum Math. (2021),  
arXiv: 2002.12715v3

Priestley dual space of the algebra  
of  $\mathbb{Z}$ -PL functions  $[0,1] \rightarrow [0,1]$   
localized at  $1/2$

## Plan

1. Priestley duality extended to double quasi-operators
2. Applying correspondence theory
3. The case of MV-algebras

# Plan

1. Priestley duality extended to double quasi-operators

→ How to dualize operations of implication type  
that respect both  $\wedge$  and  $\vee$ ?

2. Applying correspondence theory

3. The case of MV-algebras

# Plan

1. Priestley duality extended to double quasi-operators

→ How to dualize operations of implication type  
that respect both  $\wedge$  and  $\vee$ ?

2. Applying correspondence theory

→ How to dualize equational axioms?

3. The case of MV-algebras

# Plan

1. Priestley duality extended to double quasi-operators

→ How to dualize operations of implication type that respect both  $\wedge$  and  $\vee$ ?

2. Applying correspondence theory

→ How to dualize equational axioms?

3. The case of MV-algebras

→ What does this yield in the special case of MV?

# 1. Priestley duality extended to double quasi-operators

Def. Let  $L$  a bounded distributive lattice.

A double quasi-operator of reverse implication type,  
or, a "minus", is a binary function  $\ominus : L \times L^{op} \rightarrow L$

satisfying the equations:

$$\left. \begin{array}{l} - \ominus b \\ \text{is a hom.} \\ \text{except for 1} \end{array} \right\} \left\{ \begin{array}{l} (a_1 \vee a_2) \ominus b = (a_1 \ominus b) \vee (a_2 \ominus b) \\ (a_1 \wedge a_2) \ominus b = (a_1 \ominus b) \wedge (a_2 \ominus b) \\ 0 \ominus b = 0 \end{array} \right. \left. \begin{array}{l} a \ominus (b_1 \vee b_2) = (a \ominus b_1) \wedge (a \ominus b_2) \\ a \ominus (b_1 \wedge b_2) = (a \ominus b_1) \vee (a \ominus b_2) \\ a \ominus 1 = 0 \end{array} \right\} \left. \begin{array}{l} a \ominus - \\ \text{is an anti-hom} \\ \text{except for } 0 \neq 1 \end{array} \right\}$$

A pair  $(L, \ominus)$  is a  $\ominus$ -algebra.

(Throughout the talk, we moreover assume  $a \ominus 0 = a$ ,  
but this is inessential for the theory to work.)

## Theorem (Gehrke-Jónsson)

$L$  a bounded DL with Priestley space  $X$ .

Any minus operator admits two canonical liftings

$$\Theta^\sigma, \Theta^\pi : \mathcal{D}X \times (\mathcal{D}X)^{\text{op}} \rightarrow \mathcal{D}X$$

to minus operators on  $\mathcal{D}X$ ,

and any equation on  $(L, \ominus)$  lifts to both  $(\mathcal{D}X, \Theta^\sigma)$  and  $(\mathcal{D}X, \Theta^\pi)$ .

But:  $\Theta^\sigma$  respects  $\vee$ , while  $\Theta^\pi$  respects  $\wedge$  in both coordinates.

Example.  $L \models a \wedge b \leq a \ominus (a \ominus b)$   $\approx$  "MV6"

implies

$$\mathcal{D}X \models a \wedge b \leq a \ominus^\pi (a \ominus^\pi b).$$



Recall  $X \xrightarrow{\mu} \mathcal{D}X$  via  $\mu_x := X \setminus \uparrow x$ .  
 $M = m(\mu)$ .

For any  $y \in X$ ,  $- \ominus^\sigma \mu_y : \mathcal{D}X \rightarrow \mathcal{D}X$  preserves  $V$ , and  
 therefore has an upper adjoint  $- + \mu_y : \mathcal{D}X \rightarrow \mathcal{D}X$  defined by  
 $u \ominus^\sigma \mu_y \leq \sigma \iff u \leq \sigma + \mu_y$ .

$\diamond(-, -)$

$\{$

Fact. For any  $x, y \in X$ ,  $\mu_x + \mu_y \in M \cup \{1\}$ .  $+ \rightarrow R(-, -, -)$

Recall  $X \xrightarrow{v} \mathcal{D}X$  via  $v_x := \downarrow x$ .  
 $\mathcal{J} = \text{im}(v)$ .

For any  $y \in X$ ,  $- \ominus^\pi v_y : \mathcal{D}X \rightarrow \mathcal{D}X$  preserves  $\overset{\text{non-}\phi}{\wedge}$ , and

therefore has a p. lower adjoint  $- * v_y : \mathcal{D}X \rightarrow \mathcal{D}X$  defined by

$$u \ominus^\pi v_y \geq \sigma \iff (u \geq \sigma * \mu_y \text{ and } u \leq 1 \ominus^\pi v_y).$$

Fact. For any  $x, y \in X$ ,  $v_x * v_y \in \mathcal{J} \cup \{0\}$ , if  $v_x \leq 1 \ominus^\pi v_y$ .

Thus:  $\ominus^\sigma$  yields  $+$  on the dual,  
 $\ominus^\pi$  yields  $*$  on the dual.

Both  $+$  and  $*$  are partial,  
but we avoid mention  
of their domains here,  
see paper for full details.

Prop  $(*+)$   $x * y = \inf \{ x + w : w \neq y \}$

→ How does  $+$  "feel" that it is dual to a double minus?

Prop  $(+)$  For any  $x \in X$ , the translation  $x + -$  has a  
totally ordered image and an upper adjoint,  $k(x, -)$ .

Note: N. Martínez previously found the function  $x \mapsto k(x, x)$  "in the wild" in the 90s.  
(for MV-algebras)

# Theorem (Extended Priestley duality for $\ominus$ -algebras.)

The category of  $\ominus$ -algebras is dually equivalent to a category of  $\ominus$ -spaces:

$$(X, i, +, *) \quad \text{where}$$

$i$  records what  
 $a \mapsto 1 \ominus a$   
does.

$X$  a Priestley space,

$i : X^{\text{op}} \rightarrow X$  cts. order-reversing function,

$+$  :  $X^2 \rightarrow X$  upper cts. order-preserving partial fn.,

$*$  :  $X^2 \rightarrow X$  lower cts. order-preserving partial fn.,

$(*+)$  and  $(+)$  hold.

effect algebras.

$$R \subseteq X^3$$



## 2. Applying Correspondence Theory

Building on the foundation of our duality for  $\Theta$ -algebras, we may now add axioms to obtain dualities for subvarieties.

Convenient notations:  $\neg a := 1 \ominus a$   
 $a \oplus b := \neg(\neg a \ominus b)$ .

Example axioms: (1)  $\neg\neg a = a$   
(2)  $a \oplus b = b \oplus a$   
(3)  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ .  
(4)  $\oplus$  is upper residual of  $\ominus$ .

↑  
 $(a \oplus b) \ominus b \leq a$  and  $a \leq (a \ominus b) \oplus b$ .

$$(1) \neg\neg a = a$$

$$(2) a \oplus b = b \oplus a$$

$$(3) (a \oplus b) \oplus c = a \oplus (b \oplus c).$$

$$(4) \oplus \text{ is upper residual of } \ominus.$$

For each of these examples, if a  $\ominus$ -algebra  $L$  satisfies the equation (n) then so does  $(\mathcal{D}X, \ominus^\sigma)$ .

Moreover, (4) implies that  $\oplus^\pi$  coincides with  $+$  (where defined).

Now a simple and standard application of correspondence theory yields duals for these axioms.

Theorem. A  $\ominus$ -algebra  $L$  satisfies

- (1)  $\neg\neg a = a$
- (2)  $a \oplus b = b \oplus a$
- (3)  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ .
- (4)  $\oplus$  is upper residual of  $\ominus$ .

if and only if

its dual  $\ominus$ -space  $(X, i, +, *)$  satisfies

- (1)  $i^2 = i$
- (2)  $+$  commutative
- (3)  $+$  associative
- (4)  $i$  respects translation by  $+$  :  
 $x + y \leq z$  iff  $i(z) + y \leq i(x)$ .

} pre-MV-space

### 3. The case of MV-algebras

(or: the use of  $*$ )

Fact. A  $\ominus$ -algebra is (term-equivalent to) an MV-algebra iff it satisfies (1)-(4) and  $a \wedge b \leq a \ominus (a \ominus b)$  (MV6).

As we remarked in the beginning, MV6 lifts to

$$\mathcal{D}X \models u \wedge v \leq u \ominus^\pi (u \ominus^\pi v)$$

or equivalently

$$\mathcal{D}X \models \neg u \wedge v \leq (u \oplus^\pi v) \ominus^\pi u.$$

$$\forall u, v$$



Now, standard correspondence arguments let us rewrite this into

$$\forall x, y \in X, \quad v_x \leq \neg v_y \Rightarrow v_x \leq (v_x \oplus^{\pi} v_y) \ominus^{\pi} v_y.$$

To isolate the blue term, we must do:

$$\forall x, y \in X, \quad v_x \leq \neg v_y \Rightarrow v_x * v_y \leq v_x \oplus^{\pi} v_y.$$

This is where  $*$  appears.

The condition can then be further rewritten to replace  $\oplus^{\pi}$  by  $+$ ,  
using standard methods.

Theorem. The category of MV-algebras is dually equivalent to the full subcategory of  $\Theta$ -spaces on MV-spaces, i.e., pre-MV-spaces satisfying the additional (first-order!) property:

$\forall (x, y) \in \text{dom}(*), x', y' \in X$ , if  $y' \neq y$  and  $x' + y' \leq x * y$   
then  $x' \leq x$ .

"mixed cancellativity of  $*$  and  $+$ "?

## Take-aways and directions

o Extended Priestley duality with partial functions (instead of relations) on the dual is possible in the general setting of  $\ominus$ -algebras.

o Correspondence & canonicity on top of this yields concrete results for subvarieties like MV.

(Also see my first paper with Gehrke & Marra '14)

o Other example applications : - MTL-algebras (an example in the paper)  
- residuated lattices of Scott-continuous functions on complete chains

(cf. recent work with Guatto, Metcalfe & Santocchi).

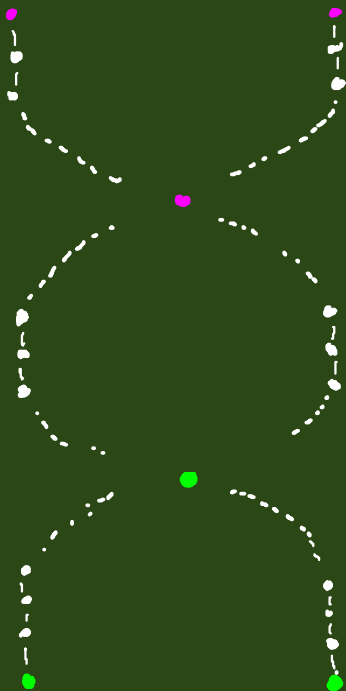
# Priestley duality for MV algebras and beyond

BLAST

June 13, 2021

Stone-Priestley session

Sam v. Gool, joint work with Wesley Fussner, Mai Gehlke and Vincenzo Marra



Priestley dual space of the algebra  
of  $\mathbb{Z}$ -PL functions  $[0,1] \rightarrow [0,1]$   
localized at  $\frac{1}{2}$

Forum Math. (2021),  
arXiv: 2002.12715v3

1. *Introduction*

2. *Background*

3. *Methodology*

4. *Results*

5. *Discussion*

6. *Conclusion*

7. *References*

8. *Appendix*

9. *Index*

10. *Index*

1. *Introduction*

2. *Methodology*

3. *Results*

4. *Discussion*

5. *Conclusion*

6. *References*

7. *Appendix*

8. *Index*

9. *Glossary*

10. *Notes*

11. *Footnotes*

12. *Endnotes*

13. *Supplementary Material*

14. *Tables*

15. *Figures*

16. *Tables of Contents*

17. *Table of Figures*

18. *Table of Tables*

19. *Table of Figures*

20. *Table of Tables*

21. *Table of Figures*

22. *Table of Tables*

23. *Table of Figures*

24. *Table of Tables*

25. *Table of Figures*

26. *Table of Tables*

27. *Table of Figures*

28. *Table of Tables*

29. *Table of Figures*

30. *Table of Tables*

31. *Table of Figures*

32. *Table of Tables*

33. *Table of Figures*

34. *Table of Tables*

35. *Table of Figures*

36. *Table of Tables*

37. *Table of Figures*

38. *Table of Tables*

39. *Table of Figures*

40. *Table of Tables*

41. *Table of Figures*

42. *Table of Tables*

43. *Table of Figures*

44. *Table of Tables*

45. *Table of Figures*

46. *Table of Tables*

47. *Table of Figures*

48. *Table of Tables*

49. *Table of Figures*

50. *Table of Tables*

51. *Table of Figures*

52. *Table of Tables*

53. *Table of Figures*

54. *Table of Tables*

55. *Table of Figures*

56. *Table of Tables*

57. *Table of Figures*

58. *Table of Tables*

59. *Table of Figures*

60. *Table of Tables*

61. *Table of Figures*

62. *Table of Tables*

63. *Table of Figures*

64. *Table of Tables*

65. *Table of Figures*

66. *Table of Tables*

67. *Table of Figures*

68. *Table of Tables*

69. *Table of Figures*

70. *Table of Tables*

71. *Table of Figures*

72. *Table of Tables*

73. *Table of Figures*

74. *Table of Tables*

75. *Table of Figures*

76. *Table of Tables*

77. *Table of Figures*

78. *Table of Tables*

79. *Table of Figures*

80. *Table of Tables*

81. *Table of Figures*

82. *Table of Tables*

83. *Table of Figures*

84. *Table of Tables*

85. *Table of Figures*

86. *Table of Tables*

87. *Table of Figures*

88. *Table of Tables*

89. *Table of Figures*

90. *Table of Tables*

91. *Table of Figures*

92. *Table of Tables*

93. *Table of Figures*

94. *Table of Tables*

95. *Table of Figures*

96. *Table of Tables*

97. *Table of Figures*

98. *Table of Tables*

99. *Table of Figures*

100. *Table of Tables*